

RELATIVE HOMOLOGICAL ALGEBRA

BY

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Introduction. The main purpose of this paper is to draw attention to certain functors, exactly analogous to the functors “Tor” and “Ext” of Cartan-Eilenberg [2], but applicable to a module theory that is relativized with respect to a given subring of the basic ring of operators. In particular, we shall show how certain relative cohomology theories for groups, rings, and Lie algebras can be subsumed under the theory of the relative Ext functor, just as (in [2]) the ordinary cohomology theories have been subsumed under the theory of the ordinary Ext functor.

Among the various relative cohomology groups that have been considered so far, some can be expressed in terms of the ordinary Ext functor; these have been studied systematically within the framework of general homological algebra by M. Auslander (to appear). A typical feature of these relative groups is that they appear naturally as terms of exact sequences whose other terms are the ordinary cohomology groups of the algebraic system in question, and of its given subsystem.

There is, however, another type of relative cohomology theory whose groups are not so intimately linked to the ordinary cohomology groups and exhibit a more individualized behaviour. Specifically, the relative cohomology groups for Lie algebras, as defined (in [3]) by Chevalley and Eilenberg, and the relative cohomology groups of groups, defined and investigated by I. T. Adamson [1], are of this second type. It is these more genuinely “relative” cohomology theories that fall in our present framework of relative homological algebra.

Our plan here is to sketch the general features of the relative Tor and Ext functors (§2) and to illustrate some of their possible uses or interpretations by a selection of unelaborated examples. Thus, §3 illustrates the use of the relative Ext functor in extending the cohomology theory for algebras. §4 deals with relative homology and relative cohomology of groups, and involves both the relative Tor functor and the relative Ext functor. §5 discusses the role played by the relative Ext functor in the cohomology theory for Lie algebras.

Since this paper is intended to serve as a preliminary survey, and since the topics dealt with are supplementary to the corresponding topics of the non-relative theory (contained in [2]), we feel justified in presupposing that the

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reader is familiar with the elementary technique and the general notions of homological algebra.

1. Relatively projective and injective modules. Let R be a ring with an identity element, 1, and let S be a subring of R containing 1. All the R -modules we shall consider are assumed to be "unitary," in the sense that 1 acts as the identity operator. An R -module will be regarded also as an S -module, in the natural way.

An exact sequence of R -homomorphisms between R -modules, $t_i: M_i \rightarrow M_{i-1}$ (or $t^i: M^i \rightarrow M^{i+1}$, with appropriate changes below) is called (R, S) -*exact* if, for each i , the kernel of t_i is a direct S -module summand of M_i . Clearly, a sequence of R -homomorphisms t_i is (R, S) -exact if and only if, for each i , (1): $t_i \circ t_{i+1} = 0$, and (2): there exists an S -homotopy, i.e., a sequence of S -homomorphisms $h_i: M_i \rightarrow M_{i+1}$ such that $t_{i+1} \circ h_i + h_{i-1} \circ t_i$ is the identity map of M_i onto itself.

An R -module A is said to be (R, S) -*injective* if, for every (R, S) -exact sequence $(0) \rightarrow U \xrightarrow{p} V \xrightarrow{q} W \rightarrow (0)$, and every R -homomorphism h of U into A , there is an R -homomorphism h' of V into A such that $h' \circ p = h$. Dually, A is said to be (R, S) -*projective* if, for every such sequence, and every R -homomorphism g of A into W , there is an R -homomorphism g' of A into V such that $g \circ g' = g^{(1)}$.

Let $t_i: M_i \rightarrow M_{i-1}$ be any (R, S) -exact sequence. It follows from our definitions and from the exactness properties of the functor Hom_R that the induced sequence of homomorphisms $\text{Hom}_R(M_{i-1}, A) \rightarrow \text{Hom}_R(M_i, A)$, $h \rightarrow h \circ t_i$, is exact whenever A is (R, S) -injective. Similarly, the induced sequence of homomorphisms $\text{Hom}_R(A, M_i) \rightarrow \text{Hom}_R(A, M_{i-1})$, $h \rightarrow t_i \circ h$, is exact whenever A is (R, S) -projective.

Let A be an S -module, and consider the group $\text{Hom}_S(R, A)$ of all S -homomorphisms of R into A , i.e., of all maps $f: R \rightarrow A$ such that $f(sr) = s \cdot f(r)$, for all $s \in S$ and all $r \in R$. This group is made into an R -module by defining, for r and r' in R , $(r \cdot f)(r') = f(r'r)$.

LEMMA 1. *For every S -module A , the R -module $\text{Hom}_S(R, A)$ is (R, S) -injective.*

Proof. Let $(0) \rightarrow U \xrightarrow{p} V \xrightarrow{q} W \rightarrow (0)$ be an (R, S) -exact sequence. Since p maps U isomorphically onto a direct S -module summand of V , it is clear that the map $h \rightarrow h \circ p$ sends $\text{Hom}_S(V, A)$ onto $\text{Hom}_S(U, A)$. On the other hand, for any R -module M , there is a natural isomorphism of the group $\text{Hom}_R(M, \text{Hom}_S(R, A))$ onto $\text{Hom}_S(M, A)$; $k \rightarrow k_1$, where $k_1(m) = k(m)(1)$, for

(¹) If ϕ is a homomorphism of a ring T into R , and $S = \phi(T)$, then the (R, S) -injective (projective) modules are precisely the ϕ -injective (projective) modules, in the sense of Cartan-Eilenberg, [2, Chap. II, §6]. In fact, the equivalence of these notions follows from Lemmas 1 and 2 below, and these lemmas themselves amount to the same as Proposition 6.3, Chap. II, of [2].

every $m \in M$. Using this, with $M = V$, and with $M = U$, we conclude that the map $k \rightarrow k \circ p$ sends $\text{Hom}_R(V, \text{Hom}_S(R, A))$ onto $\text{Hom}_R(U, \text{Hom}_S(R, A))$, which means precisely that $\text{Hom}_S(R, A)$ is (R, S) -injective.

The natural standard identification of an R -module M with the R -module $\text{Hom}_R(R, M) \subset \text{Hom}_S(R, M)$, under which each element $m \in M$ corresponds to the homomorphism m' , where $m'(r) = r \cdot m$, for every $r \in R$, gives rise to the natural exact sequence

$$(0) \rightarrow M \rightarrow \text{Hom}_S(R, M) \rightarrow \text{Hom}_S(R, M)/\text{Hom}_R(R, M) \rightarrow (0).$$

On the other hand, the map $h \rightarrow h(1)$ is an S -homomorphism $\text{Hom}_S(R, M) \rightarrow M$, sending m' back onto m . It follows that our exact sequence is actually (R, S) -exact. If M is (R, S) -injective, it follows at once that our sequence is (R, R) -exact, i.e., that $\text{Hom}_R(R, M)$ is an exact R -module summand of $\text{Hom}_S(R, M)$. Since a direct R -module summand of an (R, S) -injective module is still (R, S) -injective, we conclude that an R -module M is (R, S) -injective if and only if $\text{Hom}_R(R, M)$ is a direct R -module summand of $\text{Hom}_S(R, M)$.

PROPOSITION 1. *Let M be an (R, S) -injective R -module, and suppose that $U \rightarrow V$ is a homomorphism of R -modules such that the induced map $\text{Hom}_S(V, M) \rightarrow \text{Hom}_S(U, M)$ is an epimorphism. Then the map $\text{Hom}_R(V, M) \rightarrow \text{Hom}_R(U, M)$ is also an epimorphism.*

Proof. We have seen above that there is an R -projection of $\text{Hom}_S(R, M)$ onto $\text{Hom}_R(R, M)$. We identify $\text{Hom}_R(R, M)$ with M and note that such an R -projection induces an epimorphism $\text{Hom}_R(B, \text{Hom}_S(R, M)) \rightarrow \text{Hom}_R(B, M)$, for every R -module B . If we compose this with the natural isomorphism of $\text{Hom}_S(B, M)$ onto $\text{Hom}_R(B, \text{Hom}_S(R, M))$ (whose inverse we defined in the proof of Lemma 1) we obtain an epimorphism $\text{Hom}_S(B, M) \rightarrow \text{Hom}_R(B, M)$. Moreover, this epimorphism is natural with respect to B , in the sense of the theory of functors. Hence, using this with $B = V$ and with $B = U$, we obtain an exact and commutative diagram

$$\begin{array}{ccc} \text{Hom}_S(V, M) & \rightarrow & \text{Hom}_S(U, M) \rightarrow (0) \\ \downarrow & & \downarrow \\ \text{Hom}_R(V, M) & \rightarrow & \text{Hom}_R(U, M) \\ \downarrow & & \downarrow \\ (0) & & (0) \end{array}$$

from which it follows at once that the map $\text{Hom}_R(V, M) \rightarrow \text{Hom}_R(U, M)$ is an epimorphism.

LEMMA 2. *For every S -module A , the R -module $R \otimes_S A$ is (R, S) -projective.*

Proof. Let $(0) \rightarrow U \xrightarrow{p} V \xrightarrow{q} W \rightarrow (0)$ be an (R, S) -exact sequence. Since the kernel of q is a direct S -module summand of V , the map $h \rightarrow q \circ h$ sends

$\text{Hom}_S(A, V)$ onto $\text{Hom}_S(A, W)$. On the other hand, for any R -module M , there is a natural isomorphism of $\text{Hom}_R(R \otimes_S A, M)$ onto $\text{Hom}_S(A, M)$; $k \rightarrow k_1$, where $k_1(a) = k(1 \otimes a)$, for every $a \in A$. Using this, with $M = V$, and with $M = W$, we conclude that the map $k \rightarrow q \circ k$ sends $\text{Hom}_R(R \otimes_S A, V)$ onto $\text{Hom}_R(R \otimes_S A, W)$, which means precisely that $R \otimes_S A$ is (R, S) -projective.

If M is any R -module, the natural map $R \otimes_S M \rightarrow M$ gives rise to an exact sequence $(0) \rightarrow K_M \rightarrow R \otimes_S M \rightarrow M \rightarrow (0)$. The map $m \rightarrow 1 \otimes m$ is an S -isomorphism of M onto an S -module complement of K_M in $R \otimes_S M$, showing that our exact sequence is actually (R, S) -exact. If M is (R, S) -projective it follows that the sequence is (R, R) -exact, so that M is R -isomorphic with a direct R -module summand of $R \otimes_S M$. Since a direct R -module summand of an (R, S) -projective module is still (R, S) -projective, we may now conclude that an R -module M is (R, S) -projective if and only if it is R -isomorphic with a direct R -module summand of $R \otimes_S M$, or if and only if K_M is a direct R -module summand of $R \otimes_S M$.

PROPOSITION 2. *Let M be an (R, S) -projective R -module; and suppose that $V \rightarrow W$ is a homomorphism of R -modules such that the induced map $\text{Hom}_S(M, V) \rightarrow \text{Hom}_S(M, W)$ is an epimorphism. Then the induced map $\text{Hom}_R(M, V) \rightarrow \text{Hom}_R(M, W)$ is also an epimorphism.*

Proof. We have seen above that there is an R -isomorphism of M onto a direct R -module summand of $R \otimes_S M$. For every R -module B , this induces an epimorphism $\text{Hom}_R(R \otimes_S M, B) \rightarrow \text{Hom}_R(M, B)$. If we compose this with the natural isomorphism of $\text{Hom}_S(M, B)$ onto $\text{Hom}_R(R \otimes_S M, B)$ (whose inverse we defined in the proof of Lemma 2) we obtain an epimorphism $\text{Hom}_S(M, B) \rightarrow \text{Hom}_R(M, B)$. Moreover, this epimorphism is natural with respect to B . Hence, using this with $B = V$ and with $B = W$, we obtain an exact and commutative diagram

$$\begin{array}{ccc} \text{Hom}_S(M, V) & \rightarrow & \text{Hom}_S(M, W) \rightarrow (0) \\ \downarrow & & \downarrow \\ \text{Hom}_R(M, V) & \rightarrow & \text{Hom}_R(M, W) \\ \downarrow & & \downarrow \\ (0) & & (0) \end{array}$$

from which we see that the map $\text{Hom}_R(M, V) \rightarrow \text{Hom}_R(M, W)$ is an epimorphism.

PROPOSITION 3. *Let M be an (R, S) -projective R -module, and suppose that $U \rightarrow V$ is a homomorphism of right R -modules such that the induced map $U \otimes_S M \rightarrow V \otimes_S M$ is a monomorphism. Then the map $U \otimes_R M \rightarrow V \otimes_R M$ is also a monomorphism.*

Proof. There is an R -isomorphism of M onto a direct R -module summand

of $R \otimes_S M$. For every right R -module B , this induces a monomorphism $B \otimes_R M \rightarrow B \otimes_R (R \otimes_S M) = B \otimes_S M$. Using this, with $B = U$ and with $B = V$, we obtain an exact and commutative diagram

$$\begin{array}{ccc} (0) & & (0) \\ \downarrow & & \downarrow \\ U \otimes_R M & \rightarrow & V \otimes_R M \\ \downarrow & & \downarrow \\ (0) \rightarrow U \otimes_S M & \rightarrow & V \otimes_S M \end{array}$$

from which we see at once that the map $U \otimes_R M \rightarrow V \otimes_R M$ is a monomorphism.

2. The relative Tor and Ext functors. By an (R, S) -projective resolution of an R -module M we shall mean an (R, S) -exact sequence $\rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow (0)$ in which each C_i is (R, S) -projective. It follows at once from Lemma 2 and the remarks following its proof that every R -module has an (R, S) -projective resolution. In fact, we may take $C_0 = R \otimes_S M$, with the natural map $R \otimes_S M \rightarrow M$, and then proceed in the same way from the kernel of this map in order to obtain $C_1 \rightarrow C_0$, etc. The (R, S) -projective resolution obtained from this process is called the *standard* (R, S) -projective resolution of M .

Let B be an ordinary (left) R -module, and let A be a right R -module. Let $\cdots \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow (0)$ be the standard (right) (R, S) -projective resolution of A . Let u_i denote the R -homomorphism $X_i \rightarrow X_{i-1}$ of this resolution, for $i > 0$; but let u_0 stand for the 0-map on X_0 . It is convenient to introduce the graded R -module X whose homogeneous components of non-negative degrees are the X_i , while its homogeneous components of negative degrees are (0) . Also, denote by u the homogeneous R -endomorphism of degree -1 on X whose restriction to X_i is u_i . Thus we have a graded right R -complex (X, u) . Likewise, from the standard (R, S) -projective resolution $\rightarrow Y_1 \rightarrow Y_0 \rightarrow B \rightarrow (0)$ of B , we obtain the graded (left) R -complex (Y, v) .

Now form the graded group $Z = X \otimes_R Y$, with homogeneous components $Z_n = \sum_{i+j=n} X_i \otimes_R Y_j$, and define the homogeneous endomorphism w of degree -1 on Z such that, for $x \in X_i$ and $y \in Y_j$,

$$w(x \otimes y) = u(x) \otimes y + (-1)^i x \otimes v(y).$$

Since $u^2 = 0$ and $v^2 = 0$, it follows that $w^2 = 0$, and we have a graded group complex (Z, w) . We define $\text{Tor}_n^{(R, S)}(A, B)$ as the n th homology group $H_n(Z, w)$ of this complex. Clearly, these groups are (0) for $n < 0$.

Suppose we have two R -modules M and N , and two sequences of R -homomorphisms $\cdots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow (0)$, and $\cdots \rightarrow D_1 \rightarrow D_0 \rightarrow N \rightarrow (0)$, satisfying the following conditions: (1): in each sequence, the composite of successive maps is 0; (2): the C -sequence has an S -homotopy; (3): each D_i is (R, S) -projective. Suppose also that we are given an R -homomorphism

$q: N \rightarrow M$. Then it is easily seen that we can successively find R -homomorphisms $D_i \rightarrow C_i$ such that the resulting diagram

$$\begin{array}{ccccccc} \cdots & \rightarrow & C_1 & \rightarrow & C_0 & \rightarrow & M \rightarrow (0) \\ & & \uparrow & & \uparrow & & \uparrow q \\ \cdots & \rightarrow & D_1 & \rightarrow & D_0 & \rightarrow & N \rightarrow (0) \end{array}$$

is commutative. Moreover, if (s_i) and (t_i) are any two such systems of R -homomorphisms, there is an R -homotopy connecting them, i.e., a sequence of R -homomorphisms $h_i: D_i \rightarrow C_{i+1}$ such that, if c_i and d_i are the maps of the given sequences, $s_i - t_i = c_{i+1} \circ h_i + h_{i-1} \circ d_i$, for all i (where $h_{-1} = 0$).

This is enough information to enable one to proceed exactly as in the usual theory of Tor^R and thus establish the following facts. Every pair of R -homomorphisms $A \rightarrow A'$ and $B \rightarrow B'$ induces a unique homomorphism $\text{Tor}_n^{(R,S)}(A, B) \rightarrow \text{Tor}_n^{(R,S)}(A', B')$, and these induced homomorphisms have all the requisite properties to make $\text{Tor}_n^{(R,S)}$ a twice covariant functor. Moreover, the actual choice of the (R, S) -projective resolutions of A and B that are used in computing $\text{Tor}_n^{(R,S)}(A, B)$ has, to within natural isomorphisms, no influence on the result. In particular, one sees from this that, for $n > 0$, $\text{Tor}_n^{(R,S)}(A, B) = (0)$, whenever either A or B is (R, S) -projective. On the other hand, it can be seen directly from the definition that, for arbitrary A and B , $\text{Tor}_0^{(R,S)}(A, B) = A \otimes_R B$.

As in the case of Tor^R , it is actually sufficient to make a resolution of only one of the modules; $\text{Tor}_n^{(R,S)}(A, B)$ may be identified with the homology group $H_n(A \otimes_R Y)$ of the complex $A \otimes_R Y$ that is defined as above, using, in the place of X , the complex whose component of degree 0 is A and all whose other components are (0) . Similarly, $\text{Tor}_n^{(R,S)}(A, B) = H_n(X \otimes_R B)$. This can be proved by the methods of Chapter V of [2]. The critical fact that is needed is the following immediate consequence of Proposition 3 above: if A is (R, S) -projective then, for any (R, S) -exact sequence $\cdots \rightarrow B_i \rightarrow B_{i-1} \rightarrow \cdots$, the induced sequence $\cdots \rightarrow A \otimes_R B_i \rightarrow A \otimes_R B_{i-1} \rightarrow \cdots$ is exact (similarly, with the roles of the left and right modules interchanged). We shall briefly sketch a direct proof of the above.

Consider the subcomplex X' of X whose components are given by: $X'_i = X_i$, for $i > 0$; $X'_0 = u_1(X_1)$. The injection $X' \rightarrow X$ and the R -homomorphism $X_0 \rightarrow A$ of our resolution compose to the (R, S) -exact sequence $(0) \rightarrow X' \rightarrow X \rightarrow A \rightarrow (0)$. Since Y is (R, S) -projective, the induced sequence $(0) \rightarrow X' \otimes_R Y \rightarrow X \otimes_R Y \rightarrow A \otimes_R Y \rightarrow (0)$ is exact. Clearly, the homomorphisms of this sequence are compatible with the structure of a complex of each term. Hence if we show that the homology groups of the complex $X' \otimes_R Y$ are (0) it will follow that the map $X \otimes_R Y \rightarrow A \otimes_R Y$ induces an isomorphism of $H_n(X \otimes_R Y)$ onto $H_n(A \otimes_R Y)$, establishing the above claim concerning $\text{Tor}_n^{(R,S)}(A, B)$. In order to show that $H(X' \otimes_R Y) = (0)$ we define an increasing sequence of sub-

complexes Z^k of $X' \otimes_R Y$ whose components are given by:

$$Z_n^k = \sum_{i \leq k} X'_{n-i} \otimes_R Y_i.$$

Then each factor complex Z^k/Z^{k-1} is isomorphic with $X' \otimes_R Y_k$. Since X' has an S -homotopy and Y_k is (R, S) -projective, it follows that the homology groups of Z^k/Z^{k-1} are (0) . Hence the injections $Z^{k-1} \rightarrow Z^k$ induce isomorphisms of $H(Z^{k-1})$ onto $H(Z^k)$. Since $Z^k = (0)$, for $k < 0$, we conclude that each $H(Z^k) = (0)$. Since $H_n(Z^k) = H_n(X' \otimes_R Y)$, as soon as $k > n$, we conclude that $H(X' \otimes_R Y) = (0)$.

Let $(0) \rightarrow U \rightarrow V \rightarrow W \rightarrow (0)$ be an (R, S) -exact sequence. This leads (as above) to an exact sequence of complexes $(0) \rightarrow U \otimes_R Y \rightarrow V \otimes_R Y \rightarrow W \otimes_R Y \rightarrow (0)$. Using that $\text{Tor}_n^{(R,S)}(A, B) = H_n(A \otimes_R B)$, we obtain the exact sequence $\rightarrow \text{Tor}_n^{(R,S)}(U, B) \rightarrow \text{Tor}_n^{(R,S)}(V, B) \rightarrow \text{Tor}_n^{(R,S)}(W, B) \rightarrow \text{Tor}_{n-1}^{(R,S)}(U, B) \rightarrow$ and, clearly, there is a similar exact sequence with the roles of left and right modules interchanged.

Now we pass to the discussion of the relative Ext functor. By an (R, S) -*injective resolution* of an R -module M we shall mean an (R, S) -exact sequence $(0) \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$ in which each C^i is (R, S) -injective. It follows from Lemma 1 and the remarks following its proof that every R -module has an (R, S) -injective resolution. In fact, we may take $C^0 = \text{Hom}_S(R, M)$, with the monomorphism $M \rightarrow \text{Hom}_S(R, M)$ that we discussed in connection with Lemma 1, and proceed in the same way with the cokernel $C^0/\text{Hom}_R(R, M)$ of this monomorphism, etc. The (R, S) -injective resolution so obtained will be called the *standard* (R, S) -injective resolution of M .

Let A and B be R -modules, and let (X, u) be the R -complex obtained (as above) from the standard (R, S) -projective resolution of A . If $(0) \rightarrow B \rightarrow Y^0 \rightarrow_{v^0} Y^1 \rightarrow_{v^1} \dots$ is the standard (R, S) -injective resolution of B we denote by (Y, v) the R -complex formed by the Y^i and the maps v^i , agreeing that $Y^i = (0)$, for $i < 0$. Now form the graded group Z whose homogeneous components are the direct sums $Z^n = \sum_{i+j=n} \text{Hom}_R(X_i, Y^j)$. Define a homogeneous endomorphism w of degree 1 on Z such that, for $h \in \text{Hom}_R(X_i, Y^j)$, $w(h) = h \circ u_{i+1} + (-1)^{ij} h \circ v^j$. Then $w^2 = 0$, and (Z, w) is a graded group complex. We define $\text{Ext}_{(R,S)}^n(A, B)$ as the n th cohomology group $H^n(Z, w)$ of this complex. Clearly, these groups are (0) , for $n < 0$.

Using the result on the existence and uniqueness (up to homotopies) of maps between complexes that we have cited in our above discussion of $\text{Tor}_n^{(R,S)}$, as well as the dual result involving (R, S) -injectivity instead of (R, S) -projectivity, one can easily establish the following results, which are quite analogous to standard facts concerning Ext_R^n . Every pair of R -homomorphisms $A \rightarrow A'$, $B \rightarrow B'$ induces a unique homomorphism $\text{Ext}_{(R,S)}^n(A', B) \rightarrow \text{Ext}_{(R,S)}^n(A, B')$, and these induced homomorphisms have all the requisite properties to make $\text{Ext}_{(R,S)}^n$ a functor, contravariant in the first argument,

and covariant in the second. Moreover, the actual choice of the resolutions of A and B that are used in computing $\text{Ext}_{(R,S)}^n(A, B)$ has, to within natural isomorphisms, no influence on the result. In particular, it follows that, for $n > 0$, $\text{Ext}_{(R,S)}^n(A, B) = (0)$, whenever either A is (R, S) -projective or B is (R, S) -injective. On the other hand, it can be seen directly from the definition that, for arbitrary A and B , $\text{Ext}_{(R,S)}^0(A, B) = \text{Hom}_R(A, B)$.

Now let X' be the subcomplex of X that we have used above in reducing the computation of $\text{Tor}_n^{(R,S)}(A, B)$. Since the components of Y are (R, S) -injective, the (R, S) -exact sequence $(0) \rightarrow X' \rightarrow X \rightarrow A \rightarrow (0)$ induces an *exact* sequence of complexes $(0) \rightarrow \text{Hom}_R(A, Y) \rightarrow \text{Hom}'_R(X, Y) \rightarrow \text{Hom}'_R(X', Y) \rightarrow (0)$, where the prime in Hom' indicates that we take the weak direct sum of the homomorphism groups for the homogeneous components rather than the full group of all homomorphisms. Define a decreasing sequence of subcomplexes Z_k of $\text{Hom}'_R(X', Y)$ by taking $Z_k = \sum_{i \geq k} \text{Hom}_R(X'_{n-i}, Y^i)$. Now we can proceed in almost the same way as in our discussion of $X' \otimes_R Y$ above and conclude that the cohomology groups of the complex $\text{Hom}'_R(X', Y)$ are (0) . Hence the map $\text{Hom}_R(A, Y) \rightarrow \text{Hom}'_R(X, Y)$ induces an isomorphism of $H^n(\text{Hom}_R(A, Y))$ onto $H^n(\text{Hom}'_R(X, Y)) = \text{Ext}_{(R,S)}^n(A, B)$.

Similarly, let Y' denote the factor complex of Y modulo the image of B in Y^0 . We then have the (R, S) -exact sequence $(0) \rightarrow B \rightarrow Y \rightarrow Y' \rightarrow (0)$. Since each component of X is (R, S) -projective, the induced sequence of complexes $(0) \rightarrow \text{Hom}'_R(X, B) \rightarrow \text{Hom}'_R(X, Y) \rightarrow \text{Hom}'_R(X, Y') \rightarrow (0)$ is exact. Define a decreasing sequence of subcomplexes T_k of $\text{Hom}'_R(X, Y')$ by taking $T_k = \sum_{i \geq k} \text{Hom}_R(X_i, Y'^{n-i})$. Proceeding as before, one now shows that the cohomology groups of the complex $\text{Hom}'_R(X, Y')$ are (0) . Hence the map $\text{Hom}'_R(X, B) \rightarrow \text{Hom}'_R(X, Y)$ induces an isomorphism of $H^n(\text{Hom}'_R(X, B))$ onto $H^n(\text{Hom}'_R(X, Y))$. We may therefore identify $\text{Ext}_{(R,S)}^n(A, B)$ with $H^n(\text{Hom}'_R(X, B))$, as well as with $H^n(\text{Hom}_R(A, Y))$.

Now let $(0) \rightarrow U \rightarrow V \rightarrow W \rightarrow (0)$ be an (R, S) -exact sequence. This leads to exact sequences of complexes

$$(0) \rightarrow \text{Hom}'_R(X, U) \rightarrow \text{Hom}'_R(X, V) \rightarrow \text{Hom}'_R(X, W) \rightarrow (0),$$

and

$$(0) \rightarrow \text{Hom}_R(W, Y) \rightarrow \text{Hom}_R(V, Y) \rightarrow \text{Hom}_R(U, Y) \rightarrow (0).$$

Using our reduced expressions for $\text{Ext}_{(R,S)}^n$, we see that the first of these sequences yields an exact sequence

$$\rightarrow \text{Ext}_{(R,S)}^n(A, U) \rightarrow \text{Ext}_{(R,S)}^n(A, V) \rightarrow \text{Ext}_{(R,S)}^n(A, W) \rightarrow \text{Ext}_{(R,S)}^{n+1}(A, U) \rightarrow.$$

Similarly, the second sequence yields an exact sequence

$$\rightarrow \text{Ext}_{(R,S)}^n(W, B) \rightarrow \text{Ext}_{(R,S)}^n(V, B) \rightarrow \text{Ext}_{(R,S)}^n(U, B) \rightarrow \text{Ext}_{(R,S)}^{n+1}(W, B) \rightarrow.$$

In particular, consider the (R, S) -exact sequence $(0) \rightarrow K_A \rightarrow R \otimes_S A \rightarrow A \rightarrow (0)$. This gives rise to the exact sequence $\text{Hom}_R(R \otimes_S A, B) \rightarrow \text{Hom}_R(K_A, B) \rightarrow \text{Ext}_{(R,S)}^1(A, B) \rightarrow (0)$. For every $h \in \text{Hom}_R(K_A, B)$, let I_h denote the R -submodule of the direct sum $B + R \otimes_S A$ that consists of all elements of the form $h(k) + k$, with $k \in K_A$. Put $E_h = (B + R \otimes_S A) / I_h$. The natural map of B into E_h is a monomorphism, so that we may identify B with its image in E_h . The projection of $B + R \otimes_S A$ onto $R \otimes_S A$, followed by the natural map $R \otimes_S A \rightarrow A$, induces an R -epimorphism $E_h \rightarrow A$ whose kernel coincides with B . The map that sends each element $a \in A$ onto the coset mod I_h of $1 \otimes a \in R \otimes_S A$ is an S -monomorphism $A \rightarrow E_h$, and its composite with our epimorphism $E_h \rightarrow A$ is the identity map on A . Thus we have attached to h an S -trivial extension of the R -module B by the R -module A , i.e., an (R, S) -exact sequence $(0) \rightarrow B \rightarrow E_h \rightarrow A \rightarrow (0)$. It is not difficult to check that this yields a homomorphism of $\text{Hom}_R(K_A, B)$ onto the group (with the Baer composition) of the equivalence classes of the S -trivial extensions of B by A . Moreover, the kernel of this homomorphism turns out to be exactly the natural image of $\text{Hom}_R(R \otimes_S A, B)$ in $\text{Hom}_R(K_A, B)$. Hence we conclude that $\text{Ext}_{(R,S)}^1(A, B)$ is isomorphic with the group of the equivalence classes of the S -trivial extensions of B by A .

Finally, let us observe that, in the case where the subring S is semisimple, in the sense that every unitary S -module is semisimple (or, equivalently, that S is semisimple as an S -module), every R -exact sequence is automatically (R, S) -exact, whence we have, for all n , and all modules A, B , $\text{Tor}_n^{(R,S)}(A, B) = \text{Tor}_n^R(A, B)$, and $\text{Ext}_{(R,S)}^n(A, B) = \text{Ext}_R^n(A, B)$.

3. Relative cohomology of rings and algebras. Let K be a commutative ring, and let P be an algebra over K . We suppose that P has an identity element, 1, and that Q is a K -subalgebra of P such that $1 \in Q$. We denote by P' the K -algebra anti-isomorphic with P , and by $p \rightarrow p'$ the algebra anti-isomorphism of P onto P' . Put $R = P \otimes_K P'$, with its natural structure of an algebra over K . Let S denote the natural image of $Q \otimes_K P'$ in R . We regard P as an R -module such that, for p_1, p_2, p in P , $(p_1 \otimes p'_2) \cdot p = p_1 p p'_2$.

For any (unitary) R -module M , we define the relative cohomology group $H^n(P, Q, M)$ for (P, Q) in M as $\text{Ext}_{(R,S)}^n(P, M)$. In order to interpret and handle some of these groups, we shall give an explicit (R, S) -projective resolution of P .

For $n \geq 0$, let X_n stand for the tensor product $P \otimes_Q \cdots \otimes_Q P$, with $n+2$ factors P . The R -module structure of X_n is such that, for a, b, p_i in P , $(a \otimes b') \cdot (p_0 \otimes \cdots \otimes p_{n+1}) = a p_0 \otimes p_1 \otimes \cdots \otimes p_n \otimes p_{n+1} b$. It is easily seen that there is an R -isomorphism of $R \otimes_S X_{n-1}$ onto X_n (where, momentarily, X_{-1} is to stand for P) sending $(a \otimes b') \otimes (p_0 \otimes \cdots \otimes p_n)$ into $a \otimes p_0 \otimes \cdots \otimes p_{n-1} \otimes p_n b$, and whose inverse sends $p_0 \otimes \cdots \otimes p_{n+1}$ into $(p_0 \otimes p'_{n+1}) \otimes (p_1 \otimes \cdots \otimes p_n \otimes 1)$. Hence each X_n is (R, S) -projective. Let u_0 be the R -homomorphism of X_0 onto P such that $u_0(a \otimes b') = ab$. For $n > 0$, let u_n be

the R -homomorphism of X_n into X_{n-1} for which

$$u_n(p_0 \otimes \cdots \otimes p_{n+1}) = \sum_{i=0}^n (-1)^i (p_0 \otimes \cdots \otimes p_i p_{i+1} \otimes \cdots \otimes p_{n+1}).$$

It is easily checked that $u_{n-1} \circ u_n = 0$. In order to see that we have indeed an (R, S) -projective resolution of P , we merely have to exhibit an S -homotopy, h . Define $h_{-1}: P \rightarrow X_0$ by $h_{-1}(p) = 1 \otimes p$. For $n \geq 0$, define $h_n: X_n \rightarrow X_{n+1}$ such that $h_n(p_0 \otimes \cdots \otimes p_{n+1}) = 1 \otimes p_0 \otimes \cdots \otimes p_{n+1}$. The verification that this is actually an S -homotopy presents no difficulties.

An R -module M may be regarded as a two-sided P -module, and we shall treat it as such in our notation, whenever this is convenient. The cochain complex with the groups $\text{Hom}_R(X_n, M)$ may be identified with the cochain complex with groups $C^n(P, Q, M)$ consisting of all n -linear (with respect to K) functions f from P to M such that, for $q \in Q$ and $p_i \in P$,

$$\begin{aligned} q \cdot f(p_1, \dots, p_n) &= f(qp_1, p_2, \dots, p_n), \\ f(p_1, \dots, p_i q, p_{i+1}, \dots, p_n) &= f(p_1, \dots, p_i, qp_{i+1}, \dots, p_n), \\ f(p_1, \dots, p_n q) &= f(p_1, \dots, p_n) \cdot q. \end{aligned}$$

Here, $C^0(P, Q, M)$ is to be the group of all elements $m \in M$ for which $q \cdot m = m \cdot q$, for all $q \in Q$. The coboundary operator $\delta: C^n(P, Q, M) \rightarrow C^{n+1}(P, Q, M)$ is given by the usual formula

$$\begin{aligned} (\delta f)(p_1, \dots, p_{n+1}) &= p_1 \cdot f(p_2, \dots, p_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i f(p_1, \dots, p_i p_{i+1}, \dots, p_{n+1}) \\ &+ (-1)^{n+1} f(p_1, \dots, p_n) \cdot p_{n+1}. \end{aligned}$$

These relative cohomology groups can be used to obtain generalizations of the results of the ordinary cohomology theory for algebras (which is the case where K is a field and Q coincides with K). One of the simplest of these is as follows.

THEOREM 1. *Suppose that $H^1(P, Q, M) = (0)$, for all $P \otimes_K P'$ -modules M . Then, if a $P \otimes_K P'$ -module is semisimple for its induced structure of a $Q \otimes_K Q'$ -module, it is semisimple also as a $P \otimes_K P'$ -module.*

Proof. Let V be a $P \otimes_K P'$ -module satisfying the condition of the theorem, and let U be a submodule of V . Put $W = V/U$, and let t denote the natural epimorphism $V \rightarrow W$. Since V is $Q \otimes_K Q'$ -semisimple, there exists a $Q \otimes_K Q'$ -monomorphism $h: W \rightarrow V$ such that $t \circ h$ is the identity map on W . To every $p \in P$ let us associate a map h_p of W into U by setting $h_p(w) = h(w) \cdot p - h(w \cdot p)$. Then $h_p \in \text{Hom}_Q(W, U)$. Now $\text{Hom}_Q(W, U)$ can be given the structure of a two sided P -module such that, for $p \in P$ and $g \in \text{Hom}_Q(W, U)$, $(p \cdot g)(w) = g(w \cdot p)$ and $(g \cdot p)(w) = g(w) \cdot p$. Moreover, remembering that Q contains the

K -multiples of the identity element of P , and that our modules are unitary, it is easily verified that this actually makes $\text{Hom}_Q(W, U)$ into a $P \otimes_K P'$ -module. Now one verifies directly that the map $p \rightarrow h_p$ is an element h' of $C^1(P, Q, \text{Hom}_Q(W, U))$, and that $\delta h' = 0$. By the assumption of our theorem, there exists an element g in $C^0(P, Q, \text{Hom}_Q(W, U))$ such that $\delta g = h'$. This means that g is an element of $\text{Hom}_Q(W, U)$ such that $q \cdot g = g \cdot q$, for all $q \in Q$, and $h(w) \cdot p - h(w \cdot p) = g(w \cdot p) - g(w) \cdot p$, for all $p \in P$ and $w \in W$. Clearly, $h + g$ is therefore a $Q \otimes_K P'$ -homomorphism of W into V , and $t \circ (h + g)$ is the identity map on W . Put $f = h + g$ and, for every $p \in P$, let f_p be the map of W into U that sends each $w \in W$ into $f_p(w) = f(p \cdot w) - p \cdot f(w)$. Evidently, $f_p \in \text{Hom}_{P'}(W, U)$. We make $\text{Hom}_{P'}(W, U)$ into a two sided P -module such that, for $p \in P$ and $k \in \text{Hom}_{P'}(W, U)$, $(p \cdot k)(w) = p \cdot k(w)$ and $(k \cdot p)(w) = k(p \cdot w)$. It is easily seen that this actually makes $\text{Hom}_{P'}(W, U)$ into a $P \otimes_K P'$ -module. Now one verifies directly that the map $p \rightarrow f_p$ is an element f' of $C^1(P, Q, \text{Hom}_{P'}(W, U))$ and that $\delta f' = 0$. Using the assumption of the theorem once more, we conclude that there is an element k in $\text{Hom}_{P'}(W, U)$ such that $f(p \cdot w) - p \cdot f(w) = p \cdot k(w) - k(p \cdot w)$. Clearly, $f + k$ is a $P \otimes_K P'$ -homomorphism of W into V , and $t \circ (f + k)$ is the identity map on W . Thus U is a direct $P \otimes_K P'$ -module summand of V , and, since U is an arbitrary submodule, we have proved that V is semisimple.

Observe that, with R and S as before, the last part of our proof has actually shown that every (R, S) -exact sequence of R -modules is also (R, R) -exact. Conversely, this property implies that $H^n(P, Q, M) = (0)$, for all M and all $n > 0$, as is seen at once by applying it to an (R, S) -projective resolution of P .

In particular, the assumption of Theorem 1 is satisfied if P is the ring of all m by m matrices with elements in Q . Indeed, let p_{ij} be the usual matrix units, and let f be an element of $C^1(P, Q, M)$ such that $\delta f = 0$. Put $u = \sum_{i=1}^m p_{ii} \cdot f(p_{1i})$. Then it can be verified directly that $u \in C^0(P, Q, M)$, and that $\delta u = f$, which proves our assertion. Using also the above remark, we conclude that if P is a full matrix ring over Q then $H^n(P, Q, M) = (0)$, for all $n > 0$ and all $P \otimes_K P'$ -modules M .

We shall now obtain some general results connecting relative cohomological dimension with relative global ring dimension. These results are relativized versions of results of Cartan-Eilenberg (cf. Proposition 4 and its corollaries, in [4]) and actually represent the situation in a smoother form, in that no special assumptions need be made.

Let P, Q be as above, and let T be another K -algebra with identity element. Let A and B be $T \otimes_K P'$ -modules, written as left T - and right P -modules. We define the structure of a two sided P -module on $\text{Hom}_T(A, B)$ such that, for $a \in A$, $p \in P$, $h \in \text{Hom}_T(A, B)$, we have $(p \cdot h)(a) = h(a \cdot p)$ and $(h \cdot p)(a) = h(a) \cdot p$. It is easily checked that this actually makes $\text{Hom}_T(A, B)$ into a $P \otimes_K P'$ -module.

THEOREM 2. $H^n(P, Q, \text{Hom}_T(A, B))$ is naturally isomorphic with $\text{Ext}_{(E, F)}^n(A, B)$, where $E = T \otimes_K P'$ and F is the natural image of $T \otimes_K Q'$ in E .

For the proof we require the following lemma:

LEMMA 3. If A is (E, F) -projective then $\text{Hom}_T(A, B)$ is (R, S) -injective.

Proof. Let X and Y be R -modules (treated as two sided P -modules), and let v be an S -homomorphism of X into Y . Then v induces the homomorphism $v': \text{Hom}_{P'}(Y, B) \rightarrow \text{Hom}_{P'}(X, B)$, where $v'(h) = h \circ v$. Now $\text{Hom}_{P'}(X, B)$, and similarly the other such groups, are E -modules, with the following left T - and right P -module structures: $(t \cdot h)(x) = t \cdot h(x)$ and $(h \cdot p)(x) = h(p \cdot x)$. Clearly, v' is then a T -homomorphism. Moreover, with $q \in Q$, we have $((h \circ v) \cdot q)(x) = (h \circ v)(q \cdot x) = h(q \cdot v(x)) = ((h \cdot q) \circ v)(x)$, which shows that v' is also a right Q -homomorphism. Hence v' is an F -homomorphism. If v is not only an S -homomorphism, but even an R -homomorphism, it follows in exactly the same way that v' is an E -homomorphism.

Now let $(0) \rightarrow U \rightarrow V \rightarrow W \rightarrow (0)$ be an (R, S) -exact sequence. Then it follows from what we have just seen, applied to the homomorphisms of the given sequence, and also to those of an S -homotopy, that the induced sequence

$$(0) \rightarrow \text{Hom}_{P'}(W, B) \rightarrow \text{Hom}_{P'}(V, B) \rightarrow \text{Hom}_{P'}(U, B) \rightarrow (0) \text{ is } (E, F)\text{-exact.}$$

Next let us observe that there is a natural isomorphism between $\text{Hom}_R(U, \text{Hom}_T(A, B))$ and $\text{Hom}_E(A, \text{Hom}_{P'}(U, B))$ (and similarly with V or W in the place of U). In fact, if $h \in \text{Hom}_R(U, \text{Hom}_T(A, B))$ the corresponding element of $\text{Hom}_E(A, \text{Hom}_{P'}(U, B))$ is h' , where $h'(a)(u) = h(u)(a)$. It can be checked quite directly that the correspondence between h and h' is indeed a natural isomorphism between the two composite Hom functors.

Since A is (E, F) -projective, the epimorphism $\text{Hom}_{P'}(V, B) \rightarrow \text{Hom}_{P'}(U, B)$ of the (E, F) -exact sequence derived above induces an epimorphism $\text{Hom}_E(A, \text{Hom}_{P'}(V, B)) \rightarrow \text{Hom}_E(A, \text{Hom}_{P'}(U, B))$. Using the natural isomorphism just observed, we conclude that the map $\text{Hom}_R(V, \text{Hom}_T(A, B)) \rightarrow \text{Hom}_R(U, \text{Hom}_T(A, B))$ that is induced by the map $U \rightarrow V$ of the given (R, S) -exact sequence is an epimorphism. Thus $\text{Hom}_T(A, B)$ is (R, S) -injective, and Lemma 3 is proved.

Now let $\cdots \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow (0)$ be an (E, F) -projective resolution of A . This induces the sequence

$$(0) \rightarrow \text{Hom}_T(A, B) \rightarrow \text{Hom}_T(X_0, B) \rightarrow \text{Hom}_T(X_1, B) \rightarrow \cdots$$

Apart from the fact that the roles of (R, S) and (E, F) are interchanged, this situation is the same as that in the first part of our proof of Lemma 3. We conclude that the sequence connecting the Hom_T is (R, S) -exact, and, by Lemma 3, is therefore an (R, S) -injective resolution of $\text{Hom}_T(A, B)$. It follows that $H^n(P, Q, \text{Hom}_T(A, B))$ is naturally isomorphic with the

n th cohomology group of the complex $\text{Hom}'_R(P, \text{Hom}'_T(X, B))$. On the other hand, the map $h \rightarrow h(1)$ is easily seen to be a natural isomorphism of $\text{Hom}_R(P, \text{Hom}_T(X_i, B))$ onto $\text{Hom}_R(X_i, B)$; the inverse map is $k \rightarrow k'$, where $k'(p)(x) = k(x \cdot p) = k(x) \cdot p$. Hence the complex $\text{Hom}_R(P, \text{Hom}'_T(X, B))$ is naturally isomorphic with the complex $\text{Hom}'_R(X, B)$. This means that $H^n(P, Q, \text{Hom}_T(A, B))$ may be identified with the n th cohomology group of the complex $\text{Hom}'_R(X, B)$, i.e., with $\text{Ext}^n_{(E, F)}(A, B)$. This completes the proof of Theorem 2.

Let $c(P, Q)$ denote the largest non-negative integer n (or ∞ , if there is no largest such n) for which there exists a $P \otimes_K P'$ -module M with $H^n(P, Q, M) \neq (0)$. We call $c(P, Q)$ the *relative cohomology dimension* of the pair (P, Q) . On the other hand, let $d(P, Q)$ denote the *relative global dimension* of the pair (P, Q) , i.e., the largest non-negative integer n (or ∞ , if there is no largest such n) for which there are P -modules U and V with $\text{Ext}^n_{(P, Q)}(U, V) \neq (0)$. We shall also have to deal with $d(R, S)$, where $R = P \otimes_K P'$ and S is the natural image in R of $Q \otimes_K P'$. Here (and similarly later), where there is an evident natural map $Q \otimes_K P' \rightarrow P \otimes_K P'$ underlying our definition, we shall write $d(P \otimes_K P', Q \otimes_K P')$ instead of $d(R, S)$, whenever we wish to make a result quite explicit. Note that it follows immediately from the definitions that $c(P, Q) \leq d(P \otimes_K P', Q \otimes_K P')$. On the other hand, consider Theorem 2 with $T = P$. We have then $E = P \otimes_K P'$, while F is the natural image in $P \otimes_K P'$ of $P \otimes_K Q'$. Hence Theorem 2 shows at once that $c(P, Q) \geq d(P \otimes_K P', P \otimes_K Q')$. Thus we have $d(P \otimes_K P', Q \otimes_K P') \geq c(P, Q) \geq d(P \otimes_K P', P \otimes_K Q')$. Now interchange P and P' , and accordingly also Q and Q' . This gives $d(P' \otimes_K P, Q' \otimes_K P) \geq c(P', Q') \geq d(P' \otimes_K P, P' \otimes_K Q')$. Since switching the order of the factors in a tensor product of algebras is a natural isomorphism, we have $d(P' \otimes_K P, Q' \otimes_K P) = d(P \otimes_K P', P \otimes_K Q')$ and $d(P' \otimes_K P, P' \otimes_K Q) = d(P \otimes_K P', Q \otimes_K P')$. Hence our two chains of inequalities give equalities.

COROLLARY 1. $d(P \otimes_K P', Q \otimes_K P') = c(P, Q) = d(P \otimes_K P', P \otimes_K Q') = c(P', Q')$.

Now consider Theorem 2 with $T = K$. Then $E = P'$ and $F = Q'$, whence we see that $c(P, Q) \geq d(P', Q')$. Similarly, $c(P', Q') \geq d(P, Q)$.

COROLLARY 2. $c(P, Q) \geq \max(d(P, Q), d(P', Q'))$.

We are now in a position to obtain the following generalization of a result due to Cartan-Eilenberg [2, Chapter IX, Theorem 7.11].

THEOREM 3. *Let Q be a commutative ring with identity element, and let $P = Q[x_1, \dots, x_n]$ be the ring of polynomials in n variables with coefficients in Q . Then $d(P, Q) = n = c(P, Q)$.*

Proof. We regard Q as a P -module such that, for $q \in Q$ and $p(x_1, \dots, x_n) \in P$, we have $p(x_1, \dots, x_n) \cdot q = p(0, \dots, 0)q$. We shall use a complex that was originally introduced by J. L. Koszul [6]: let V denote the free Q -module

of rank n , with the free basis $y_1, \dots, y_n$. Let V_k denote the homogeneous component of degree k of the exterior Q -algebra constructed over V . Thus, in particular, $V_0 = Q$, $V_1 = V$, and $V_k = (0)$, for all $k > n$. The homogeneous component of degree k of Koszul's complex is defined as $T_k = P \otimes_Q V_k$. Let t_0 denote the map $p(x_1, \dots, x_n) \rightarrow p(0, \dots, 0)$ of P onto Q . Clearly, t_0 is a P -epimorphism $P = T_0 \rightarrow Q$. Let $v \rightarrow v'$ denote the Q -monomorphism $V \rightarrow P$ that sends each y_i onto x_i . For $k > 0$, let t_k be the P -homomorphism of T_k into T_{k-1} that sends $p \otimes v_1 \cdots v_k$, where $p \in P$ and $v_i \in V$, onto

$$\sum_{i=1}^n (-1)^{i-1} p v'_i \otimes v_1 \cdots v_{i-1} v_{i+1} \cdots v_k.$$

It is easily verified that $t_{k-1} \circ t_k = 0$. Since each T_k is (P, Q) -projective (actually, P -free), there remains only to exhibit an S -homotopy in order to conclude that (T, t) is a (P, Q) -projective resolution of Q . We define $h_{-1}: Q \rightarrow T_0$ by putting $h_{-1}(q) = 1 \otimes q$. Now observe that each T_k is a free Q -module; for $k > 0$, the elements $1 \otimes y_{j_1} \cdots y_{j_k}$ and $x_{i_1} \cdots x_{i_q} \otimes y_{j_1} \cdots y_{j_k}$, where $j_1 < j_2 < \cdots < j_k$ and $i_1 \leq i_2 \leq \cdots \leq i_q$, constitute a free Q -basis for T_k ; and the elements $1 \otimes 1$ and $x_{i_1} \cdots x_{i_q} \otimes 1$ constitute a free Q -basis for T_0 . Hence we can define an S -homomorphism $h_k: T_k \rightarrow T_{k+1}$ such that $h_0(1 \otimes 1) = 0$, $h_0(x_{i_1} \cdots x_{i_q} \otimes 1) = x_{i_1} \cdots x_{i_{q-1}} \otimes y_{i_q}$, and, for $k > 0$,

$$h_k(1 \otimes y_{i_1} \cdots y_{i_k}) = 0,$$

while

$$h_k(x_{i_1} \cdots x_{i_q} \otimes y_{j_1} \cdots y_{j_k}) = (-1)^k x_{i_1} \cdots x_{i_{q-1}} \otimes y_{j_1} \cdots y_{j_k} y_{i_q},$$

if $i_q > j_k$, and $= 0$ otherwise. It is not difficult to verify directly that h is indeed a Q -homotopy of (T, t) .

Since Q is annihilated by every polynomial with constant term 0, it follows immediately from the definition of t_k that the induced map $\text{Hom}_P(T_{k-1}, Q) \rightarrow \text{Hom}_P(T_k, Q)$ is the 0-map, for all $k > 0$. Hence, for all $k \geq 0$, we may identify $\text{Ext}_{(P, Q)}^k(Q, Q)$ with $\text{Hom}_P(T_k, Q)$. In particular, we see that $\text{Ext}_{(P, Q)}^n(Q, Q)$ is isomorphic with Q . Hence we have $d(P, Q) \geq n$.

Now let $R = P[z_1, \dots, z_n]$ be the polynomial ring in n new variables with coefficients in P . There is an isomorphism χ of R onto $P \otimes_Q P$ such that $\chi(p) = 1 \otimes p$, for $p \in P$, and $\chi(z_i) = x_i \otimes 1 - 1 \otimes x_i$. Then $\chi(P) = 1 \otimes P$, and the natural operations of $P \otimes_Q P (= P \otimes_Q P')$ on P make P into an R -module (via χ) such that $r(z_1, \dots, z_n) \cdot p = r(0, \dots, 0)p$, for all $p \in P$ and all $r(z_1, \dots, z_n) \in R$. Now consider the construction of the beginning of this proof, with R in the place of P , and P in the place of Q . It gives an (R, P) -projective resolution of P whose components reduce to (0) , for degree $> n$. Through the isomorphism χ , this becomes a $(P \otimes_Q P', Q \otimes_Q P')$ -projective resolution of P . Hence we conclude that $c(P, Q) \leq n$. Using our first result and Corollary 2, we obtain $n \leq d(P, Q) \leq c(P, Q) \leq n$, and Theorem 3 is proved.

4. Relative homology and cohomology of groups. Let G be a group, and let K be a subgroup of G . Let R be the group ring $Z(G)$ of G over the ring Z of the integers, and let $S = Z(K)$. If $r \rightarrow r'$ is the coefficient sum homomorphism of $Z(G)$ onto Z we regard Z as an R -module such that $r \cdot z = r'z$. For a unitary R -module M , we define the relative cohomology groups for (G, K) in M by $H^n(G, K, M) = \text{Ext}_{(R,S)}^n(Z, M)$. Accordingly, we shall exhibit an (R, S) -projective resolution of Z .

For all $n \geq 0$, let X_n denote the free abelian group generated by the set of $(n+1)$ -tuples $(A_0, \dots, A_n)$ of cosets $A_i = g_i K$, with $g_i \in G$. The R -module structure of X_n is such that, for $g \in G$, $g \cdot (A_0, \dots, A_n) = (gA_0, \dots, gA_n)$. Agreeing momentarily that $X_{-1} = Z$, there is an R -isomorphism of $R \otimes_S X_{n-1}$ onto X_n sending $g \otimes (A_1, \dots, A_n)$ onto $(gK, gA_1, \dots, gA_n)$; its inverse sends $(A_0, \dots, A_n)$ onto $g_0 \otimes (g_0^{-1}A_1, \dots, g_0^{-1}A_n)$, where g_0 is any representative of A_0 in G , whose particular choice has no influence on the image. Hence each X_n is (R, S) -projective.

We define u_0 to be the coefficient sum homomorphism of X_0 onto Z . For $n > 0$, we define $u_n: X_n \rightarrow X_{n-1}$ such that

$$u_n(A_0, \dots, A_n) = \sum_{i=0}^n (-1)^i (A_0, \dots, A_{i-1}, A_{i+1}, \dots, A_n).$$

As is well known, (X, u) is an acyclic R -complex, and there remains only to exhibit an S -homotopy, h . We define $h_{-1}: Z \rightarrow X_0$ by putting $h_{-1}(z) = z(K)$. For $n \geq 0$, we define $h_n: X_n \rightarrow X_{n+1}$ such that $h_n(A_0, \dots, A_n) = (K, A_0, \dots, A_n)$. The verification that h is then indeed an S -homotopy presents no difficulties.

If M is an R -module, the cochain complex $\text{Hom}'_R(X, M)$ is the usual homogeneous group complex, except for the fact that cosets have taken the place of group elements.

Any R -module M may be regarded also as a right R -module such that, for $g \in G$ and $m \in M$, the right transform $m \cdot g$ of m by g is given by $g^{-1} \cdot m$. With this being understood, we define the n th relative homology group $H_n(G, K, M)$ as $\text{Tor}_n^{(R,S)}(M, Z)$.

In the case where K is of finite index in G , one can proceed as in the ordinary cohomology theory for finite groups (cf. [2]) and introduce negative dimensional cohomology groups in such a way that (after $H^0(G, K, M)$ has been replaced by a certain factor group of itself), for every (R, S) -exact module sequence $(0) \rightarrow U \rightarrow V \rightarrow W \rightarrow (0)$, one has a doubly infinite exact cohomology sequence

$$\dots \rightarrow H^q(G, K, U) \rightarrow H^q(G, K, V) \rightarrow H^q(G, K, W) \rightarrow H^{q+1}(G, K, U) \rightarrow \dots$$

In fact, for all $n \geq 0$, let $X_{-n-1} = \text{Hom}_Z(X_n, Z)$, made into an R -module such that, for $h \in \text{Hom}_Z(X_n, Z)$, $g \in G$ and $x \in X_n$, $(g \cdot h)(x) = h(g^{-1} \cdot x)$. Since (X, u) is (R, S) -exact and $Z \subset S$, the induced sequence

$(0) \rightarrow Z = \text{Hom}_Z(Z, Z) \rightarrow \text{Hom}_Z(X_0, Z) = X_{-1} \rightarrow X_{-2} \rightarrow \cdots$ is (R, S) -exact.

Composing this sequence with the given resolution (X, u) of Z , we obtain a commutative and (R, S) -exact diagram

$$\begin{array}{ccccccc}
 \cdots & \rightarrow & X_1 & \rightarrow & X_0 & \rightarrow & X_{-1} \rightarrow X_{-2} \rightarrow \cdots \\
 & & & & \searrow & & \nearrow \\
 & & & & & Z & \\
 & & \nearrow & & & \searrow & \\
 (0) & & & & & & (0)
 \end{array}$$

Using the natural isomorphism between $\text{Hom}_R(U, \text{Hom}_Z(X, Z))$ and $\text{Hom}_R(X, \text{Hom}_Z(U, Z))$, for arbitrary R -modules U and X , we see easily that the R -modules $\text{Hom}_Z(X_n, Z)$ ($n \geq 0$) are (R, S) -injective⁽²⁾. This has the significance that, if we make the same construction with any other (R, S) -projective resolution (X', u') of Z , the resulting "complete resolution," in which $X'_{-n-1} = \text{Hom}_Z(X'_n, Z)$, is equivalent, up to R -homotopies, with the one displayed above.

If M is an R -module, we define the generalized cohomology groups $H^q(G, K, M)$, for $-\infty < q < \infty$, as the q th cohomology groups of the complex $\text{Hom}'_R(X, M)$, where X now stands for the weak direct sum of *all* the X_q , with $-\infty < q < \infty$. It follows from the remark above that these cohomology groups are still independent of the particular choice of the original (R, S) -projective resolution of Z .

Now let us make use of some of the special features of the resolution (X, u) that we defined in the beginning. It is clear from our definition that (for $n \geq 0$) each X_n has a finite (because K is of finite index in G) free basis over Z whose elements are permuted among themselves under the action of G . It follows from this that $\text{Hom}_Z(X_n, Z)$ is also (R, S) -projective. In fact, using such a basis, we can define an R -isomorphism of X_n onto $\text{Hom}_Z(X_n, Z)$. This is done as follows: for each basis element x of X_n , let x' be the element of $\text{Hom}_Z(X_n, Z)$ that takes the value 1 at x , but the value 0 at each of the other basis elements. Clearly, there is a Z -isomorphism of X_n onto $\text{Hom}_Z(X_n, Z)$ that maps each basis element x onto x' . Since the basis elements are permuted among themselves under the action of G , we have $(g \cdot x)' = g \cdot x'$, for every $g \in G$ and every basis element x . Hence our Z -isomorphism is actually an R -isomorphism.

Hence our above (R, S) -exact sequence connecting the X_q for $-\infty < q < \infty$ is a complete (R, S) -projective resolution of Z , and therefore leads to the doubly infinite exact cohomology sequences announced above. It is easily seen from our definition of the X_n that the cohomology groups $H^q(G, K, M)$ are precisely those defined by Adamson in [1]. Moreover, Adamson has shown that the doubly infinite exact cohomology sequence results from any

⁽²⁾ The proof is similar to, but much simpler than that of Lemma 3.

R -exact sequence $(0) \rightarrow U \rightarrow V \rightarrow W \rightarrow (0)$ with the property that, for every subgroup L of K that is an intersection of conjugates of K , the induced sequence $(0) \rightarrow U^L \rightarrow V^L \rightarrow W^L \rightarrow (0)$ of the L -fixed subgroups is still exact. Since this condition is evidently satisfied whenever the given sequence is (R, S) -exact, Adamson's result is stronger than what we have obtained above by using merely the general theory.

Next we observe that, if K is of finite index in G , we can augment the system of homology groups for (G, K) by using the same complete resolutions of Z . Thus, for all $-\infty < q < \infty$, we define $H_q(P, Q, M)$ as the q th homology group of the complex whose components are the $M \otimes_R X_q$. This replaces the former $H_0(P, Q, M)$ by a certain subgroup of itself. If $(0) \rightarrow U \rightarrow V \rightarrow W \rightarrow (0)$ is an (R, S) -exact sequence, we have then a doubly infinite exact homology sequence

$$\cdots \rightarrow H_q(G, K, U) \rightarrow H_q(G, K, V) \rightarrow H_q(G, K, W) \rightarrow H_{q-1}(G, K, U) \rightarrow \cdots$$

In the ordinary theory for finite groups, one has $H^{-n}(G, M) \approx H_{n-1}(G, M)$ and $H_{-n}(G, M) \approx H^{n-1}(G, M)$, for all $n > 1$ (also for $n = 1$, when the modified H^0 and H_0 are used). The reason for this is that then the groups X_n of the usual homogeneous resolution of Z (take $K = (1)$, in the above) have a finite free Z -basis (the $(n+1)$ -tuples of group elements) on which G operates without fixed points, which means that the X_n are free R -modules of finite rank. Now it is easily seen that a finite free R -basis for X_n allows us to define, for every R -module M , isomorphisms between $\text{Hom}_R(\text{Hom}_Z(X_n, Z), M)$ and $M \otimes_R X_n$, and between $M \otimes_R \text{Hom}_Z(X_n, Z)$ and $\text{Hom}_R(X_n, M)$, these isomorphisms leading to isomorphisms between the homology and cohomology groups, as indicated above.

In the relative case, this breaks down, and we are left with separate (augmented) homology and cohomology structures. There is a very simple example showing that the positive dimensional homology groups differ from the negative dimensional cohomology groups, and also that the negative dimensional homology groups differ from the positive dimensional cohomology groups.

Let $G = K \times L$, where K and L are groups of order 2. Let s be a generator for K , t a generator for L . Let Z' be the R -module whose underlying group is the additive group Z of the integers and where the operators are given by $s \cdot z = -z$, and $t \cdot z = z$, for all $z \in Z'$. Then the K -fixed subgroup Z'^K of our R -module Z' is (0) . Now it has been shown by Adamson (see also §6, here) that if K is a normal subgroup of G one has $H^q(G, K, M) = H^q(G/K, M^K)$, for every R -module M . Hence we have $H^q(G, K, Z') = (0)$. On the other hand, let $I(K)$ be the kernel of the coefficient sum homomorphism of $Z(K)$ onto Z . As we shall see in §6, we have then $H_q(G, K, M) = H_q(G/K, M/I(K) \cdot M)$. By what we have said above about the ordinary homology theory for finite groups, this last group may be identified with $H^{-q-1}(G/K, M/I(K) \cdot M)$. If

$M = Z'$, we have $M/I(K) \cdot M = Z/2Z$, with trivial G/K -operators. G/K being cyclic, these cohomology groups are well known. In our case, they all coincide with $Z/2Z$.

We refer to [1] for such information on the relative cohomology groups as is significant for field theory. The relative homology groups have not been investigated, and their significance remains to be elucidated.

By way of illustration, we shall be content here with an elementary interpretation of $H^2(G, K, M)$ in the framework of the theory of group extensions. Let M be an R -module. By a (G, K) -extension of M we shall mean a pair (E, p) , where E is a group containing both K and M , and p is a homomorphism of E onto G satisfying the following conditions: (1): for $e \in E$ and $m \in M$, $eme^{-1} = p(e) \cdot m$; (2): p is the identity map on K ; (3): the kernel of p is M ; (4): the identity map of K into E can be extended to a map q of G into E such that $p \circ q$ is the identity map on G , and $q(gk) = q(g)k$, $q(kg) = kq(g)$, for all $g \in G$ and $k \in K$.

We shall say that two such extensions (E, p) and (E', p') are equivalent if there exists an isomorphism t of E onto E' which is the identity map on M and on K , and such that $p' \circ t = p$. With the usual composition, one obtains the structure of an abelian group on the set of equivalence classes of (G, K) -extensions of M . We shall show that this group is canonically isomorphic with $H^2(G, K, M)$.

We shall employ a shortened form of the cochain complex derived from the homogeneous (R, S) -projective resolution of Z . At the beginning of this section, we exhibited an R -isomorphism of $R \otimes_S X_{n-1}$ onto X_n . This yields an isomorphism of $\text{Hom}_R(X_n, M)$ onto $\text{Hom}_R(R \otimes_S X_{n-1}, M)$. Now this last group may evidently be identified with $\text{Hom}_S(X_{n-1}, M)$, whose elements may be regarded as the functions f of n -tuples of cosets gK , taking values in M , and such that $f(kA_1, \dots, kA_n) = k \cdot f(A_1, \dots, A_n)$, whenever $k \in K$. In terms of these functions, the coboundary operator turns out to be given by

$$(\delta f)(A_1, \dots, A_{n+1}) = g_1 \cdot f(g_1^{-1} A_2, \dots, g_1^{-1} A_{n+1}) \\ + \sum_{i=1}^{n+1} (-1)^i f(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_{n+1}),$$

where g_1 is any representative in G of the coset A_1 . Here, a 0-cochain is an element m of M^K , and $(\delta m)(A_1) = g_1 \cdot m - m$.

Now a (G, K) -extension of M gives rise to 2-cocycles f of the type just described, as follows: $f(A_1, A_2) = q(g_1)q(g_1^{-1}g_2)q(g_2)^{-1}$, where g_i is a representative of A_i in G , whose particular choice has no influence on the expression defining $f(A_1, A_2)$. If q' is any other map satisfying the above condition (4), then $q'(g) = r(g)q(g)$, with $r(g) \in M$. Also, for $k \in K$, we have $kr(g)q(g) = kq'(g) = q'(kg) = r(kg)kq(g)$, whence $r(kg) = k \cdot r(g)$. On the other hand, $r(g)q(g)k = q'(g)k = q'(gk) = r(gk)q(g)k$, whence $r(gk) = r(g)$. Hence r may be regarded

as a 1-cochain r' for (G, K) in M . If f' is defined from q' in the same way as f was defined from q we find that $f' = f + \delta r'$. Hence a (G, K) -extension of M determines a unique element of $H^2(G, K, M)$. As in the analogous interpretation of the ordinary cohomology group $H^2(G, M)$, it can now be shown that this correspondence induces an isomorphism between $H^2(G, K, M)$ and the group of equivalence classes of the (G, K) -extensions of M .

5. Relative cohomology for Lie algebras. Let L be a Lie algebra over a field F , and let K be a subalgebra of L . Denote the universal enveloping algebras of K and L by S and R , respectively, identifying S with its canonical image in R . If M is an L -module, over F , then the structure of a unitary R -module is induced on M in the natural way, according to the definition of the universal enveloping algebra. We regard F as a unitary R -module with trivial operators, i.e., such that $L \cdot F = (0)$. Now we define the relative cohomology groups for (L, K) in M as the groups $\text{Ext}_{(R, S)}^n(F, M)$.

We shall give an explicit (R, S) -projective resolution of F . Let R^+ denote the ideal of R that is generated by the elements of L , and similarly define S^+ . We take $X_0 = R/R^+$, noting that this R -module is isomorphic with $R \otimes_S F$, and thus (R, S) -projective. The canonical projection of R onto F induces an R -homomorphism u_0 of X_0 onto F . For $n > 0$, we put $X_n = R \otimes_S \cdots \otimes_S R \otimes_S R^+/R^+$, with n factors R . We then define the R -homomorphism $u_n: X_n \rightarrow X_{n-1}$ such that, for $r_i \in R$ and $v \in R^+/R^+$,

$$u_n(r_1 \otimes \cdots \otimes r_n \otimes v) = \sum_{i=1}^{n-1} (-1)^{i-1} r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n \otimes v \\ + (-1)^{n-1} r_1 \otimes \cdots \otimes r_{n-1} \otimes r_n \cdot v.$$

It is easy to check that $u_{n-1} \circ u_n = 0$, and since each X_n is (R, S) -projective, there remains only to exhibit an S -homotopy, h . We define $h_{-1}: F \rightarrow R/R^+$ as the restriction to F of the natural R -homomorphism $R \rightarrow R/R^+$. For $n > 0$, we define $h_n: X_n \rightarrow X_{n+1}$ such that $h_n(r_1 \otimes \cdots \otimes r_n \otimes v) = 1 \otimes r_1 \otimes \cdots \otimes r_n \otimes v$. Finally, we define $h_0: X_0 \rightarrow X_1$ as follows: the canonical projection $r \rightarrow r^+$ of R onto R^+ , followed by the natural R -homomorphism $R^+ \rightarrow R^+/R^+$, induces an F -linear map $v \rightarrow v^+$ of R/R^+ onto R^+/R^+ . We set $h_0(v) = 1 \otimes v^+$. Then h_0 is indeed an S -homomorphism. For, let r be a representative in R of $v \in R/R^+$, and let $s \in S$. Then $s \cdot v$ is the coset mod RS^+ of sr . Hence $(s \cdot v)^+$ is the coset mod RS^+ of $(sr)^+$. But $(sr)^+ = sr^+ + r^0 s^+$, where r^0 is the component of r in F . Hence $(s \cdot v)^+$ is the coset mod RS^+ of sr^+ , and therefore coincides with $s \cdot v^+$. Hence h_0 is an S -homomorphism. Clearly, the other h_n are also S -homomorphisms, and it is easy to verify that h is indeed an S -homotopy.

In order to compare these new relative cohomology groups with those defined by Chevalley-Eilenberg (see [3], and [5]), we have to consider an altogether different R -complex. This complex arises from a relativization of the R -projective resolution of F that is used in showing that the ordinary

cohomology groups $H^n(L, M)$ coincide with the groups $\text{Ext}_R^n(F, M)^{(3)}$. This resolution is as follows: denote by $E_n(L)$ the homogeneous component of degree n of the exterior algebra built over the F -space L . The adjoint representation of L extends to an L -module structure on $E_n(L)$ such that, for x, x_i in L , $x \cdot (x_1 \cdots x_n) = [x, x_1]x_2 \cdots x_n + \cdots + x_1 \cdots x_{n-1}[x, x_n]$. We put $X_n = R \otimes_F E_n(L)$. On the one hand, X_n has then its natural R -module structure, derived from the ordinary multiplications in R . On the other hand, using the L -module structure of $E_n(L)$, we can define an L -module structure on X_n whose operations commute with the operations of the natural R -module structure. In fact, for $x \in L$, we define the operator t_x on X_n such that, for $r \in R$ and $v \in E_n(L)$, $t_x(r \otimes v) = r \otimes x \cdot v - rx \otimes v$.

The R -homomorphisms u_i of the resolution of F will be R -homomorphisms with respect to the natural R -module structure, in principle. However, they will also be R -homomorphisms with respect to the R -module structure that corresponds to our additional L -module structure. The definition of the u_i is inductive. The map u_0 is the canonical projection of R onto F . The map $u_1: X_1 \rightarrow X_0$ is defined such that, for $r \in R$ and $x \in L = E_1(L)$, $u_1(r \otimes x) = rx$. We have then evidently $u_0 \circ u_1 = 0$, and it is easily checked that each t_x , and each operator of the natural R -module structure, commutes with u_0 and u_1 . Now suppose that u_k has already been defined for $k = 0, 1, \cdots, n-1$, where $n \geq 2$, and that, for all $2 \leq k \leq n-1$, $r \in R$, $x \in L$ and $v \in E_{k-1}(L)$, we have $u_k(r \otimes vx) = u_{k-1}(r \otimes v)x + (-1)^k t_x(r \otimes v)$ (the meaning of the notation in the first term is that, if $u_{k-1}(r \otimes v) = \sum_i r_i \otimes v_i$, then $u_{k-1}(r \otimes v)x$ stands for $\sum_i r_i \otimes v_i x$). Then we attempt to define u_n in such a way that this relation continues to hold for $k = n$. Clearly, this requirement can be met, provided only that the expression on the right depends only on r and the *product* $vx \in E_n(L)$, but not *separately* on v and x . By the multilinearity of the expression, and by the definition of the exterior algebra as a homomorphic image of the tensor algebra, this will follow as soon as we have shown that the expression reduces to 0 whenever v is of the form wx , with $w \in E_{n-2}(L)$. For $n = 2$, the expression then becomes $rw x \otimes x + t_x(r \otimes wx) = rw x \otimes x - rx \otimes wx = 0$, observing that, here, $w \in F$. Now suppose that $n > 2$. Then, reducing the first term, our expression can be written $(u_{n-2}(r \otimes w)x + (-1)^{n-1} t_x(r \otimes w))x + (-1)^n t_x(r \otimes wx) = (-1)^{n-1} t_x(r \otimes w)x + (-1)^n t_x(r \otimes w)x = 0$. Hence u_n can indeed be defined as desired. Evidently, u_n is then an R -homomorphism with respect to the natural R -module structure. Also, by induction on n , one shows first that u_n commutes with each t_x , and then that $u_{n-1} \circ u_n = 0$. We omit this straightforward computation.

Now let J be the ideal of $E(L)$ that is generated by the elements of K , and put $J_n = J \cap E_n(L)$. Let Q_n denote the kernel of the natural homomorphism $R \otimes_F E_n(L) \rightarrow R \otimes_S E_n(L)$ (note that, through its L -module structure, $E_n(L)$

⁽³⁾ This is carried out in [2, Chap. XIII]; our treatment includes the ordinary groups as a special case.

has the structure of an R -module; our tensor product is taken with respect to the induced S -module structure of $E_n(L)$). Put $Y_n = R \otimes_F J_n + Q_n$. We shall show that $u_n(Y_n) \subset Y_{n-1}$. Evidently, $t_z(X_n) \subset Q_n$, for every $z \in K$. On the other hand, it is seen at once by induction on k that, with $z_i \in K$, $r \in R$ and $v \in E_n(L)$, we have $rz_1 \cdots z_k \otimes v - r \otimes z_1 \cdots z_k \cdot v \in t_K(X_n) = \sum_{z \in K} t_z(X_n)$. Hence $Q_n = t_K(X_n)$. Since u_n commutes with the operators t_z , it follows from this that $u_n(Q_n) \subset Q_{n-1}$. On the other hand, the inductive definition of u_n shows at once that $u_n(R \otimes_F J_n) \subset R \otimes_F J_{n-1} + t_K(X_{n-1})$, whence we may conclude that $u_n(Y_n) \subset Y_{n-1}$. Hence u_n induces an R -homomorphism q_n of $Z_n = X_n/Y_n$ into $Z_{n-1} = X_{n-1}/Y_{n-1}$ such that $q_{n-1} \circ q_n = 0$. The S -module structure of $E_n(L)$ induces in the natural fashion an S -module structure on $E_n(L/K)$, and we may evidently identify Z_n with $R \otimes_S E_n(L/K)$, concluding, in particular, that each Z_n is (R, S) -projective.

Next we show that the complex of the Z_n is acyclic. For this purpose, we define an increasing sequence of subcomplexes, as follows: let R_p denote the set of all elements of R that can be written in the form $\sum_i r_i s_i$, with $s_i \in S$, and r_i a product of no more than p elements of L . In particular, $R_0 = S$, and $R_p = (0)$, for $p < 0$. Put $Z_n^p = R_{p-n} \otimes_S E_n(L/K)$ ⁽⁴⁾. It follows at once from the inductive definition of u_n that $q_n(Z_n^p) \subset Z_{n-1}^p$, so that the Z_n^p ($n = 0, 1, \dots$) make up a subcomplex Z^p of the complex of the Z_n . Now consider the factor complex Z^p/Z^{p-1} . It is easily seen that $(Z^p/Z^{p-1})_n$ is isomorphic with $P_{p-n}(L/K) \otimes_F E_n(L/K)$, where $P_t(L/K)$ denotes the homogeneous component of degree t of the symmetric algebra built over L/K . The boundary operator $(Z^p/Z^{p-1})_n \rightarrow (Z^p/Z^{p-1})_{n-1}$ that is induced by q_n is thereby transported into an F -linear map of $P_{p-n}(L/K) \otimes_F E_n(L/K)$ into $P_{p-n+1}(L/K) \otimes_F E_{n-1}(L/K)$ sending each element of the form $a \otimes v_1 \cdots v_n$ onto $\sum_{i=1}^n (-1)^{i-1} a v_i \otimes v_1 \cdots v_{i-1} v_{i+1} \cdots v_n$, as is seen by induction on n . Thus, the complex Z^p/Z^{p-1} no longer involves the Lie algebra structure. Moreover, we know from the proof of Theorem 3 that this complex is acyclic. Hence the injections $Z^{p-1} \rightarrow Z^p$ induce isomorphisms of $H(Z^{p-1})$ onto $H(Z^p)$. Hence each $H(Z^p) = (0)$, and it follows that $H(Z) = (0)$.

If M is an R -module, the cohomology groups of the complex with the cochain groups $\text{Hom}_R(Z_n, M)$ evidently coincide with the relative cohomology groups for (L, K) in M as defined by Chevalley-Eilenberg in [3] (see also [5]). If we knew that the complex (Z, q) has an S -homotopy, we could therefore conclude that these relative groups coincide with the groups $\text{Ext}_{(R,S)}^n(F, M)$. In general, the question of the existence of an S -homotopy, and also that of the coincidence of the Chevalley-Eilenberg groups with the $\text{Ext}_{(R,S)}^n(F, M)$ is not settled. However, in the case where the relative groups have a topological interpretation, as the topological cohomology groups, over the field of the

⁽⁴⁾ R has an F -basis consisting of the ordered monomials in the elements of an ordered basis of L , an endsection of which is a basis of K . Hence each R_p is a direct right S -module summand of B . This is used implicitly from now on.

real numbers, of homogeneous spaces of compact analytic groups, it is possible to prove the existence of an S -homotopy for our complex (Z, q) . The algebraic essentials of this case consist in the base field being of characteristic 0, and the subalgebra K being reductive in L , i.e., L being semisimple as a K -module. We then have the following result:

THEOREM 4. *Let L be a finite dimensional Lie algebra over a field F of characteristic 0. Let K be a subalgebra of L , and assume that K is reductive in L . Then the complex (Z, q) defined above is an (R, S) -projective resolution of F , and hence the relative cohomology groups for (L, K) in any L -module M (as defined by Chevalley-Eilenberg) coincide with the groups $\text{Ext}_{(R, S)}^n(F, M)$.*

Proof. There remains only to show that the complex (Z, q) has an S -homotopy. Since K is reductive in L , L/K is semisimple for its natural structure as a K -module. It follows (see Proposition 1 of [5]) that the n -fold tensor power of L/K is also semisimple as a K -module. Since $E_n(L/K)$ is a homomorphic image of this K -module, it is semisimple. On the other hand, R may be regarded as an L -module such that, for $z \in L$ and $r \in R$, $z \cdot r = zr - rz$. Let R'_p be the subspace of R that is spanned by the products of no more than p elements of L . Then, with respect to the L -module structure just defined, R'_p is an L -submodule of R . Moreover, as an L -module, R'_p is a homomorphic image of the sum of the homogeneous components of degrees p of the tensor algebra built over L , with its natural structure as an L -module. Restricting the operator algebra to K , we obtain the structure of a semisimple K -module on this tensor algebra, because K is reductive in L . Hence also R'_p is a semisimple K -module. Since R is the sum of the submodules R'_p , it follows that R is semisimple as a K -module, and, in fact, is the sum of finite dimensional simple K -submodules. The same is therefore true for the tensor product $R \otimes_F E_n(L/K)$ (again by Proposition 1 of [5]), when this is regarded as a K -module with operators a_z ($z \in K$) such that $a_z(r \otimes u) = (zr - rz) \otimes u + r \otimes z \cdot u$. Now it is evident that each a_z maps the above Y_n into themselves. Hence the a_z induce the structure of a K -module on Z_n , for which Z_n is semisimple. We have $a_z(r \otimes u) = zr \otimes u + t_z(r \otimes u)$, and, since $t_z(X_n) \subset Y_n$, it follows that the K -module structure on Z_n that is induced by the a_z coincides with the K -module structure that corresponds to the natural S -module structure of Z_n . Hence Z_n is semisimple with respect to its natural S -module structure. Since the complex (Z, q) is acyclic, it follows that it has an S -homotopy, and Theorem 4 is proved.

6. Reduction of relative groups. If K is a normal subgroup of a group G , and M is a G -module, then the relative cohomology groups for (G, K) in M reduce to the ordinary cohomology groups for G/K in the K -fixed part M^K of M . An analogous reduction takes place in the case of a Lie algebra L relative to an ideal K of L . We shall absorb these known results in a more general result concerning certain (R, S) -projective resolutions.

Let R be a ring with 1, and let P be a central subring of R such that $1 \in P$. Let S be a subring of R such that $P \subset S$. Suppose also that there is given a projection homomorphism q of R onto P . Let S_q denote the kernel of the restriction of q to S . Assume that $S_q R \subset RS_q$, i.e., that the left ideal RS_q of R is also a right ideal. We may then form the factor ring $R' = R/RS_q$. We assume, finally, that R' is P -projective, for its natural structure as a P -algebra.

Using the projection q , we may regard P as an R -module such that, for $p \in P$ and $r \in R$, $r \cdot p = q(r)p$. Now consider the standard (R, S) -projective resolution (X, u) of P . One sees at once by induction on n that each X_n is annihilated by S_q , so that it may also be regarded as an R' -module. We claim that, when so regarded as an R' -complex, (X, u) is an R' -projective resolution of P .

In order to see this, consider $R \otimes_S U$, where U is any R -module that is annihilated by S_q . The exact double R -module sequence $(0) \rightarrow RS_q \rightarrow R \rightarrow R' \rightarrow (0)$ induces the exact R -module sequence $RS_q \otimes_S U \rightarrow R \otimes_S U \rightarrow R' \otimes_S U \rightarrow (0)$. Since U is annihilated by S_q , the first map of this sequence is the 0-map. Hence $R \otimes_S U$ is isomorphic with $R' \otimes_S U$. Now R' and U are both annihilated by S_q , and S/S_q is isomorphic with P in such a way that corresponding elements operate in the same way on U, R' . Hence we have $R' \otimes_S U = R' \otimes_P U$. Now suppose, furthermore, that U is P -projective. Then U is a direct P -module summand of a free P -module, N , say, and $R' \otimes_P U$ is a direct R' -module summand of $R' \otimes_P N$, which is a free R' -module. Hence $R' \otimes_P U$ is R' -projective, and therefore also $R \otimes_S U$, when regarded as an R' -module in the natural fashion, is R' -projective.

Since P is annihilated by S_q and is P -projective, it follows therefore that $X_0 (= R \otimes_S P)$ is annihilated by S_q , and R' -projective. Since R' is P -projective, every R' -projective module is also P -projective. Since the complex (X, u) has an S -homotopy, which is a fortiori a P -homotopy, the kernel of u_0 is a direct P -module summand of X_0 , and hence is still P -projective. Hence we may now repeat the whole argument to conclude that X_1 is R' -projective, and that the kernel of u_1 is P -projective, etc. Hence (X, u) is indeed an R' -projective resolution of P .

Now let M be an R -module, and let M' denote the set of all elements of M that are annihilated by S_q . Then M' is an R -submodule of M . Moreover, M' may be regarded as an R' -module and, as such, is canonically isomorphic with $\text{Hom}_R(R', M)$, where R' is regarded as a left R -module and a right R' -module in the natural fashion. If U is any R' -module, regarded also as an R -module via the natural homomorphism of R onto R' , there is a natural isomorphism between $\text{Hom}_{R'}(U, \text{Hom}_R(R', M))$ and $\text{Hom}_R(U, M)$ sending $h \in \text{Hom}_{R'}(U, \text{Hom}_R(R', M))$ onto $h_1 \in \text{Hom}_R(U, M)$, where $h_1(a) = h(a)(1)$, for every $a \in U$; the inverse sends k onto k' , where $k'(a)(r') = k(r' \cdot a)$. Using these isomorphisms, for $U = X_i$ ($i = 0, 1, \dots$), and the isomorphism between

M' and $\text{Hom}_R(R', M)$, we find that $\text{Ext}_{(R,S)}^n(P, M)$ is naturally isomorphic with $\text{Ext}_{R'}^n(P, M')$.

In the application to the cohomology of groups, we have $P=Z$, and q is the coefficient sum homomorphism of $R=Z(G)$ onto Z . If K is a normal subgroup of G , we take $S=Z(K)$ and see at once that all our above assumptions hold in this case. The result is that $H^n(G, K, M)$ is naturally isomorphic with $H^n(G/K, M^K)$. Moreover, if K is of finite index in G , it is easily seen that the R -modules $\text{Hom}_Z(X_n, Z)$, as defined in §4, may be regarded also as R' -modules and, as such, are the same as those resulting when the X_n are regarded as R' -modules and the operations on $\text{Hom}_Z(X_n, Z)$ are defined by referring to G/K . Hence the result for the cohomology groups holds also in negative dimensions.

Similarly, if M is a right R -module, and U is an R' -module as before, we may identify $M \otimes_R U$ with $M \otimes_R R' \otimes_{R'} U$. Using this with $U=X_i$ ($i=0, 1, \dots$), we find that $\text{Tor}_n^{(R,S)}(M, P)$ is naturally isomorphic with $\text{Tor}_n^{R'}(M \otimes_R R', P)$. Moreover, $M \otimes_R R'$ may evidently be identified with $M/M \cdot S_q$.

In the case of homology for groups, this shows that, if K is normal in G , $H_n(G, K, M)$ is naturally isomorphic with $H_n(G/K, M/I(K) \cdot M)$, where $I(K)$ is as in §4. As above, this continues to hold for negative n , when K is of finite index in G .

In the application to the cohomology of Lie algebras, we have $P=F$ (the base field), and q is the projection of the universal enveloping algebra R of the Lie algebra L onto F . If K is an ideal in L , we take S to be the universal enveloping algebra of K , and see that all the above assumptions now hold. The result is that then $H^n(L, K, M)$ is naturally isomorphic with $H^n(L/K, M^K)$, where now M^K is the set of all elements of M that are annihilated by K . This result holds for the relative groups as proposed here, as well as for the relative groups of Chevalley-Eilenberg (in which case it is well known, of course).

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